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Coaxial Line with Helical Inner Conductor*

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TO OBTAIN an approximate solution for the fields in a coaxial line with a helical inner conductor, the helix is replaced by a fictitious surface that is conducting only in the helix direction, an approximation used in the early work on traveling-wave tubes. Maxwell's equations are solved for the lowest "mode" (all fields independent of angle) when the medium inside the helix has permittivity and permeability different from that of the medium surrounding the helix. Equations for the velocity along the axis, characteristic impedance, attenuation constant, and Q are given.

The significant parameter is $(2\pi Na)(2\pi a/\lambda)$, where N = number of turns per unit length, a = helix radius, and λ = wavelength. When this parameter is considerably less than 1, the velocity and characteristic impedance depend only on the dimensions. The dielectric inside the helix has only a second-order effect, while the dielectric outside the helix has a first-order effect. The wave appears to propagate along the helix wire with the velocity of light only when the outer conductor is very close to the helix; as the outer-conductor diameter is increased, the apparent velocity along the wire gradually increases and reaches a limiting value when the outer conductor is infinitely large. For the shapes generally used, the apparent velocity along the wire is rarely more than 30-percent greater than the velocity of light, but with an infinitely large outer conductor this velocity can be 2 or 3 times the velocity of light.

When $(2\pi Na)(2\pi a/\lambda)$ is greater than 1, the wave appears to propagate along the helix wire with the velocity of light, and the characteristic impedance depends only on the ratio of wavelength to helix radius. Introducing higher-dielectric-constant material inside or outside of the helix has a first-order effect, and the effect is the same whether the material is inside or outside the helix.

The outer-conductor loss is appreciable, about one-fourth to one-half the helix loss for the usual shapes with low velocities along the axis. The unloaded Q's can be about the same as with conventional coaxial lines. The Q does not depend on the length of the resonator if $(2\pi Na)(2\pi a/\lambda)$ is less than about 0.5, so that for frequencies below about 500 megacycles per second, the volume is considerably less than the volume of a conventional coaxial-line resonator. A few measurements to check the formulas for Q are presented.

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Coaxial lines with helical inner conductors are used in many applications—in traveling-wave tubes, as delay lines, high-Q resonators, and high-characteristic-impedance transmission lines, and in extending microwave impedance-matching techniques to frequencies as low as 300 kilocycles. An analysis of this transmission line is given in this paper. The equations reduce to those previously published when A) the number of turns per unit length goes to zero (the standard coaxial line) and B) the outer conductor is removed (the helix used in traveling-wave tubes). The assumption that the electromagnetic wave travels along the helix wire with a velocity very close to that of light is not true except in extreme cases so that formulas based on this assumption may be in error.

Maxwell's equations are solved by replacing the helix with a fictitious surface that is conducting only in the helix direction. This method has been successfully used in the early work on traveling-wave tubes.[1,2] All fields are assumed to be independent of angle (the lowest "mode"). These assumptions have been questioned on

* Reprinted from *Proceedings of the IRE*, volume 42, pages 1315–1319; August, 1954.

[1] J. R. Pierce, "Traveling-Wave Tubes," D. Van Nostrand Company, Incorporated, New York, New York; 1950: page 229. See also J. R. Pierce, "Theory of Beam-Type Traveling-Wave Tubes," *Proceedings of the IRE*, volume 35, pages 111–123; February, 1947.

[2] L. L. Chu and J. D. Jackson, "Field Theory of Traveling-Wave Tubes," *Proceedings of the IRE*, volume 36, pages 853–863; July, 1948.

sound theoretical grounds, but the results given here are good enough for most engineering applications. The details of the derivation are given in the appendix.

After this paper was written, a translation of a paper[3] originally published in Russian was obtained. It presents a general solution of the problem, emphasizing the high-frequency behavior with an electron beam. There is no discussion or derivation of characteristic impedance, velocity, or Q, the main points of this paper. Another paper[4] gives an expression for the velocity and an approximate expression for the characteristic impedance.

1. *Velocity*

When the dielectrics on both sides of the helix are identical, the velocity along the axis is given by

$$(c/V)^2 = 1 + (M\lambda/2\pi a)^2, \qquad (1)$$

[3] L. N. Loshakov, "Propagation of Waves along a Coaxial Spiral Line in the Presence of an Electron Beam," *Journal of Technical Physics*, volume 19, pages 578–595; May, 1949.

[4] C. O. Lund, "Broadband Transition from Coaxial Line to Helix," *RCA Review*, volume 11, pages 133–142; March, 1950.

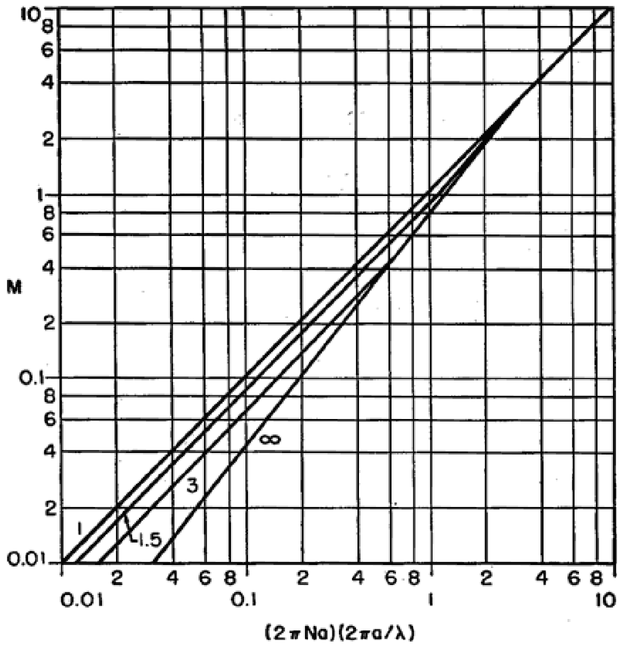

Figure 1—M plotted against $(2\pi Na)$ $(2\pi a/\lambda)$ for the indicated values of b/a.

$$(2\pi Na)\,\frac{2\pi a}{\lambda} = M\,\frac{J_0(jM)}{J_1(jM)}\left[\frac{\dfrac{H_0^{(1)}(jMb/a)}{J_0(jMb/a)} - \dfrac{H_0^{(1)}(jM)}{J_0(jM)}}{\dfrac{H_1^{(1)}(jMb/a)}{J_1(jMb/a)} - \dfrac{H_1^{(1)}(jM)}{J_1(jM)}}\right]^{1/2}. \qquad (2)$$

All symbols are defined in section 5.

These equations can be solved by plotting $(2\pi Na)(2\pi a/\lambda)$ against M, with b/a as a parameter as shown in Figure 1 for $\epsilon_1 = \epsilon_2$. Then if N, a, and λ are known, the velocity can be obtained from the first equation.

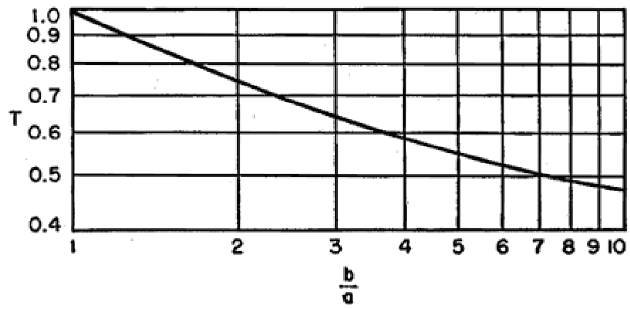

Figure 2—T in (3) is a function of b/a.

When the diameters are small $(b/\lambda_0 < V/2\pi c)$, the above equations simplify to

$$\left.\begin{aligned}\left(\frac{c}{V}\right)^2 &\approx 1 + \frac{[1-(a/b)^2](2\pi Na)^2}{2\ln(b/a)}\\ &= 1 + T^2(2\pi Na)^2.\end{aligned}\right\}(3)$$

T versus b/a is plotted in Figure 2. Usually $(2\pi Na)^2$ is large, so that

$$\left.\begin{aligned}c/V &\approx T(2\pi Na)\\ &\approx (2\pi Na)(a/b)^{1/2}.\end{aligned}\right\}(4)$$

The last approximation becomes inaccurate for large values of b/a. The factor T is, for $(2\pi Na) \gg 1$, the ratio of the velocity of light to the velocity of the wave along the wire. This ratio does not depart markedly from 1 for most practical cases, but in the case of helical antennas[5] this ratio can be larger than 2.

When the diameters are large $(b/\lambda_0 > V/c)$, the equations reduce to

$$(c/V)^2 \approx 1 + (2\pi Na)^2 = 1/\sin^2\psi. \qquad (5)$$

[5] A. G. Kandoian and W. Sichak, "Wide-Frequency-Range Tuned Helical Antennas and Circuits," *Electrical Communication*, volume 30, pages 294–299; December, 1953: also, *Convention Record of the 1953 IRE National Convention*, volume 1, pages 42–45.

For this case (usually encountered in traveling-wave tubes), the wave travels along the wire with the velocity of light. When the diameters are small and the dielectric inside the helix is different from the dielectric outside the helix, the velocity is very nearly the same as if the dielectric were uniform throughout. On the other hand, for large diameters,

$$\frac{c_2}{V} \approx (2\pi Na)\left(\frac{\epsilon_1+\epsilon_2}{2}\right)^{1/2}. \qquad (6)$$

This case is interesting because the velocity is the same whether the dielectric is inside the helix or outside of it and, in addition, does not depend on the ratio of diameters.

2. *Characteristic Impedance*

The voltage is the integral of E_{r2} from $r = a$ to $r = b$. The current is given by

$$I = 2\pi a H_{\theta 2}(r = a) - j\omega\epsilon_1 \int_0^a E_{x1}(2\pi r)dr,$$

$$Z_0 = \frac{c}{V} 30\pi J_0^2(jM)\left|\frac{H_0^{(1)}(jMb/a)}{J_0(jMb/a)} - \frac{H_0^{(1)}(jM)}{J_0(jM)}\right|. \qquad (7)$$

Figure 3 shows how the characteristic impedance varies with M and b/a for $\epsilon_1 = \epsilon_2$.

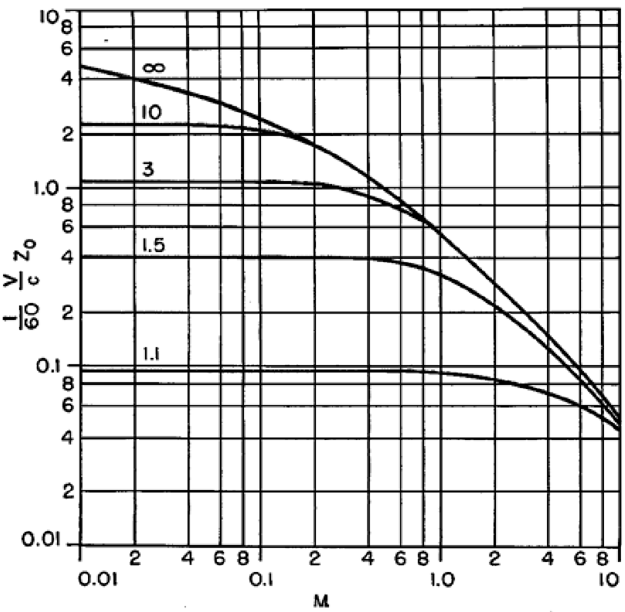

Figure 3—Relation of M to (1/60) $(V/c)Z_0$ for indicated values of b/a.

When the diameters are small, this equation reduces to

$$Z_0 \approx \frac{c_2}{V}\left(\frac{\epsilon_0}{\epsilon_2}\right)^{1/2} 60 \ln\left(\frac{b}{a}\right). \qquad (8)$$

Using (3), this can be written

$$\left.\begin{aligned} Z_0 &\approx 120\pi Na\left[\frac{(1 - a^2/b^2)\ln(b/a)}{2}\right]^{1/2}\left(\frac{\epsilon_0}{\epsilon_2}\right)^{1/2} \\ &= 120\pi NaT\,(\epsilon_0/\epsilon_2)^{1/2}\ln(b/a). \end{aligned}\right\} (9)$$

As Winkler[6] has shown, the maximum value when holding N and a constant is obtained with $b/a = 2.06$.

This approximation holds when the curves in Figure 3 have zero slope. For the usual values of b/a (1.5 or greater), M must be less than about 0.4.

For $b/a = \infty$ and $\epsilon_1 = \epsilon_2$, the characteristic impedance is

$$Z_0 = (c/V)30\pi J_0(jM)H_0^{(1)}(jM). \qquad (10)$$

For $M < 0.5$,

$$Z_0 \approx 60\frac{c}{V}\ln\frac{1.12}{M}. \qquad (11)$$

For $M > 0.5$,

$$Z_0 \approx \frac{c}{V}\frac{30}{M} \approx \frac{30\lambda}{2\pi a}. \qquad (12)$$

The last approximation holds when $c/V \gg 1$.

The standard formula for the inductance of a long solenoid can be obtained by treating the solenoid as a short length of short-circuited line and using (10) and (29).

3. *Losses and Q*

3.1 Outer-Conductor Losses

The power loss in the outer conductor can be calculated since the tangential magnetic field is known.[7] The power lost per square meter of surface is

$$P_L = \pi b(H_x^2 + H_\theta^2)Z_{\mathrm{wall}}. \qquad (13)$$

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[6] M. R. Winkler, Discussion on "High Impedance Cable," by H. E. Kallman, *Proceedings of the IRE*, volume 35, page 1097; October, 1947.

[7] R. I. Sarbacher and W. A. Edson, "Hyper- and Ultra-High Frequency Engineering," John Wiley and Sons, Incorporated, New York, New York; 1943: page 262.

The attenuation constant is

$$\alpha = \frac{P_L}{2P_{\text{inc}}} = \frac{P_L}{2I^2Z_0}. \tag{14}$$

For small diameters and high velocity ratios (the usual case of interest),

$$H_x \gg H_\theta,$$

$$I \approx 2\pi a H_{\theta 2}(r = a), \tag{15}$$

$$\alpha \approx \frac{Z_{\text{wall}}}{4\pi b Z_0}\left(\frac{a}{b}\right)^2 (2\pi Na)^2. \tag{16}$$

A result similar to (16) has been obtained by Bogle.[8] The factor $Z_{\text{wall}}/(4\pi b Z_0)$ is the attenuation constant due to the outer conductor in an ordinary coaxial line.

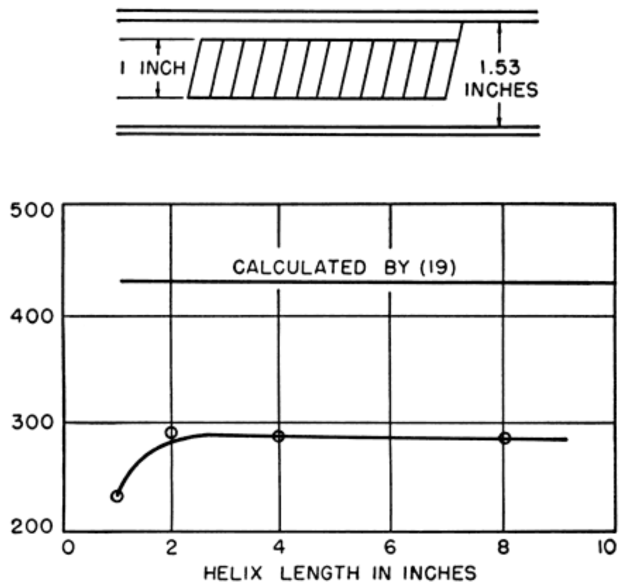

Figure 4—Experimental and calculated values of Q versus helix length. In the experimental helix, $Nd = 1/3$ and there were 116 turns with one end connected to the outer conductor.

3.2 Helix Losses

The procedure used to calculate the outer-conductor losses leads to erroneous results when applied to the helix because the boundary conditions of the helix are not satisfied exactly. An approximate result can be obtained by assuming that the resistance of the helix is the same as an isolated conductor of the same diameter.

$$\alpha \approx \frac{Z_{\text{wall}}(2\pi Na)}{4\pi d Z_0} = \frac{Z_{\text{wall}}(2\pi Na)^2}{8\pi^2 a Z_0 (Nd)}. \tag{17}$$

The factor $(2\pi Na)$ is introduced because the unit of length is taken along the axis rather than along the helix wire.

This approximation does not take into account the proximity effect (apparent increase of resistance in a conductor due to the proximity of other conductors).

3.3 The Q

The unloaded Q using copper conductors is given by[9]

$$Q = \beta/2\alpha. \tag{18}$$

For small diameters, this becomes

$$Q \approx \frac{120\pi a(1 - a^2/b^2) f_{mc}^{1/2}}{\dfrac{1}{2\pi Nd} + \left(\dfrac{a}{b}\right)^3}. \tag{19}$$

This equation indicates that the optimum value of (Nd) is one as large as possible due to the approximations used in deriving these equations. However, it is known from experimental data that the optimum value of (Nd) is between 0.3 and 0.4.

The optimum value of b/a depends on (Nd). Using $(Nd) = 0.35$ and $b/a = 2.23$, the Q is

$$Q_{\text{opt}} \approx 250 b f_{mc}^{1/2}. \tag{20}$$

For a coaxial line with $b/a = 3.6$

$$Q_{\text{opt}} = 210 b f_{mc}^{1/2}. \tag{21}$$

These optimum Q's have been derived without regard to the resulting volume, subject only to the restriction that $(2\pi Na)(2\pi a/\lambda)$ be less than 0.5–1.0. The quantity *Q/volume* is a maximum for $(Nd) = 0.35$ when $b/a = 1.57$.

$$Q_{\text{opt}} \approx 200 b f_{mc}^{1/2}. \tag{22}$$

4. Experimental Results

According to (19), the Q does not depend on the length or volume of the resonator. Measurements were made on a series of resonators with the same helix and outer-conductor diameters but with different wire diameters and turns per

[8] A. G. Bogle, "Effective Inductance and Resistance of Screened Coils," *Journal of the Institution of Electrical Engineers*, volume 87, pages 299–316; September, 1940.

[9] "Reference Data for Radio Engineers," Federal Telephone and Radio Company, New York, New York, third edition; 1949: pages 304–319.

inch in the helix, but holding the product (Nd) constant. The coils ranged in length from 1 inch to 8 inches (2.5 to 20 centimeters). The resonant frequencies were near 9 megacycles. Q versus helix length is shown in Figure 4. Over most of the range, the Q is constant and equal to about two-thirds of the calculated values. Some of the discrepancy between measurement and theory is due to the use of commercial copper, to the method of soldering the helix to the outer conductor, and to the use of a solid polystyrene form for winding the helices. The shortest helix has a characteristic impedance less than the others (because $(2\pi Na)(2\pi a/\lambda)$ is about 1), so that its Q should be smaller.

5. Glossary

a = mean radius of helix

b = outer-conductor radius

$B = \dfrac{p_1 \tan \psi J_0(p_1a)}{j\omega\mu_1 J_1(p_1a)}$

c = velocity of light

d = radius of helix wire

$D = \dfrac{J_0(p_1a)}{-GJ_0(p_2a) + N_0(p_2a)}$

E_x,
E_r,
E_θ = electric field in the indicated direction

f = frequency

f_{mc} = frequency in megacycles

$F = \dfrac{p_2 \tan \psi J_0(p_1a)}{j\omega\mu_2[-HJ_1(p_2a) + N_1(p_2a)]}$

$G = \dfrac{N_0(p_2b)}{J_0(p_2b)}$

$H = \dfrac{N_1(p_2b)}{J_1(p_2b)}$

H_x,
H_r,
H_θ = magnetic field in the indicated direction

$H_0^{(1)}$,
$H_1^{(1)}$ = Hankel functions of the first kind

I = current

J_0,
J_1 = Bessel functions of the first kind

$M = |pa|$

N = number of turns per unit length

N_0,
N_1 = Bessel functions of the second kind

$p^2 = \gamma^2 + \omega^2\epsilon\mu$

P_{inc} = incident power

Q = quality factor

V = velocity along the axis

Z_0 = characteristic impedance

$Z_{\text{wall}} = (\pi f\mu/\sigma)^{1/2}$

α = attenuation constant

$\beta = 2\pi/\lambda_a$ = phase constant along the axis

$\beta_0 = 2\pi/\lambda$ = free-space phase constant

$\gamma = \alpha + j\beta$ = propagation constant

ϵ = permittivity

λ = free-space wavelength

μ = permeability

σ = conductivity

$\omega = 2\pi \times$ frequency

6. Appendix—Fields and Propagation Constants

The coordinate system is shown in Figure 5. All conductors and dielectrics are assumed loss-

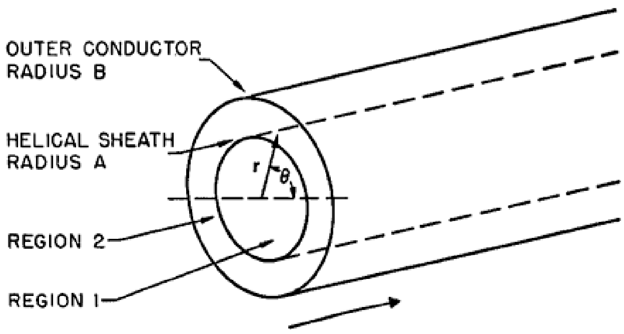

Figure 5—Coordinate system. ψ is the angle between a turn of the helix and the cross-sectional plane of the helix.

less and the fields are independent of θ. Inside the helix, the longitudinal fields are taken to be of the form

$$\left.\begin{aligned} E_{x1} &= J_0(p_1r), \\ H_{x1} &= BJ_0(p_1r). \end{aligned}\right\} \quad (23)$$

Bessel functions of the second kind are not used because they go to infinity at $r = 0$.

Between the helix and the outer conductor, the longitudinal fields are taken to be of the form

$$\left.\begin{aligned} E_{z2} &= D[-GJ_0(p_2r) + N_0(p_2r)], \\ H_{z2} &= F[-HJ_0(p_2r) + N_0(p_2r)]. \end{aligned}\right\} \quad (24)$$

The boundary conditions $E_{z2} = 0$ and $\partial H_{z2}/\partial r = 0$ at $r = b$ have been applied in the above equations.

By using Maxwell's equations [10] the remaining fields can be determined.

Inside the helix ($r \leqslant a$),

$$\left.\begin{aligned} E_{r1} &= \frac{\gamma_1}{p_1} J_1(p_1r), \\ E_{\theta 1} &\doteq \frac{(-)Bj\omega\mu_1}{p_1} J_1(p_1r), \\ H_{r1} &= \frac{B\gamma_1}{p_1} J_1(p_1r), \\ H_{\theta 1} &= \frac{j\omega\epsilon_1}{p_1} J_1(p_1r). \end{aligned}\right\} \quad (25)$$

Outside the helix ($a \leqslant r \leqslant b$),

$$\left.\begin{aligned} E_{r2} &= \frac{D\gamma_2}{p_2}[-GJ_1(p_2r) + N_1(p_2r)], \\ E_{\theta 2} &= \frac{-Fj\omega\mu_2}{p_2}[-HJ_1(p_2r) + N_1(p_2r)], \\ H_{r2} &= \frac{F\gamma_2}{p_2}[-HJ_1(p_2r) + N_1(p_2r)], \\ H_{\theta 2} &= \frac{Dj\omega\epsilon_2}{p_2}[-GJ_1(p_2r) + N_1(p_2r)]. \end{aligned}\right\} \quad (26)$$

The factor $\exp[j(\omega t - \gamma x)]$ multiplying the right-hand sides of all of the above equations has been omitted.

[10] See page 243 of footnote reference 7.

The boundary conditions at the helical sheet ($r = a$) are

$$\left.\begin{aligned} E_{z1} &= E_{z2}, \\ E_{\theta 1} &= E_{\theta 2}, \\ \frac{E_{z1}}{E_{\theta 1}} &= \frac{E_{z2}}{E_{\theta 2}} = -\cot\psi, \\ (H_{z1} - H_{z2}) &+ (H_{\theta 1} - H_{\theta 2})\cot\psi = 0. \end{aligned}\right\} \quad (27)$$

For a nontrivial (that is, one in which the coefficients B, D, and F are not zero) solution to exist, the following equation must hold:

$$\begin{aligned} &\omega^2 \frac{\mu^2}{p^2} \cot^2\psi \\ &\times\left\{\frac{\epsilon_2}{p_2}\left[\frac{-GJ_1(p_2a)+N_1(p_2a)}{-GJ_0(p_2a)+N_0(p_2a)} - \frac{\epsilon_1 J_1(p_1a)}{p_1 J_0(p_1a)}\right]\right\} \\ &= -\left(\frac{\mu_2}{\mu_1}\right)\left(\frac{p_1}{p_2}\right)\frac{J_0(p_1a)}{J_1(p_1a)} \\ &\qquad + \left[\frac{-HJ_0(p_2a)+N_0(p_2a)}{-HJ_1(p_2a)+N_1(p_2a)}\right]. \end{aligned} \quad (28)$$

This equation determines the propagation constant when the dimensions, dielectric constants, et cetera, are specified. When the outer conductor is removed, the equation reduces to that given by Harris.[11] When the dielectric on both sides of the helix is identical and the outer conductor is removed, the equation becomes

$$\frac{\beta_0^2 \cot^2\psi}{p^2} = -\frac{J_0(pa)H_0^{(1)}(pa)}{J_1(pa)H_1^{(1)}(pa)}. \quad (29)$$

This result is the same as that given by Pierce,[1] and by Chu and Jackson.[2]

[11] L. A. Harris, H. R. Johnson, A. Karp, and L. D. Smullin, "Some Measurements of Phase Velocity along a Helix with Dielectric Supports," Massachusetts Institute of Technology Radiation Laboratory Report 93: January 21, 1949.