An introduction to the art of Solenoid Inductance and Impedance Calculation

Part II

Solenoid Impedance and Q

By David W Knight[1]

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Overview

1 Ottery St Mary, Devon, England. http://g3ynh.info/
2 Revision history.
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By David W Knight

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Note
References cited on multiple occasions are given an alias at the first occurrence, as indicated in [square brackets].

2.1. Introduction

This is the second part of an article on the subject of solenoid inductance and impedance calculation. Part 1 gives a general introduction, but concentrates on the calculation of inductance from purely magnetic considerations. Such calculations have traditionally been said to lead to a quantity known as the 'low-frequency' inductance', but the approach adopted in this work is that of separating internal and external inductance (where internal inductance is due to the magnetic energy stored in the body of the conductor), so that the frequency dependence of the internal inductance can be taken into account. Thus the magnetic calculation no longer leads to 'low-frequency' inductance, but to a frequency dependent inductance calculated from magnetic considerations. This corresponds to the '*equivalent lumped inductance*', L, of the coil on the assumption that the current distribution along the length of the conductor is uniform.

The length-wise uniform current requirement[3] is satisfied, to a good approximation, in coils used in electrical circuits. Hence, the calculated pure inductance, L, provides the basis for calculating coil impedance. This leads to another type of inductance, the apparent inductance, L', which is defined as the reactance of the series-form inductor impedance divided by the angular frequency, 2πf. To find that impedance, we need L as a parameter, but we also need to calculate the AC resistance of the coil and its self-capacitance. Note that apparent inductance is positive when the coil is operating below its self-resonant frequency (SRF), swaps from positive to negative at the SRF, and alternates thereafter as we sweep through a series of self-resonance overtones. Lumped element theory is generally concerned with the operating region below the SRF. The ratio of reactance to AC resistance, an important parameter in the design of filters, is trivially extracted from the impedance calculation and is called the 'Q' (quality or goodness) of the coil.

Since the calculation of lumped inductance and self capacitance are dealt with in separate documents, this article is primarily concerned with the matter of determining AC resistance and combining it with the other quantities to obtain the impedance and Q.

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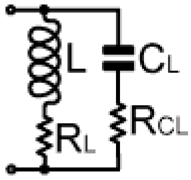
3 The current is not uinform over the wire cross-section, due to the skin and proximity effects.

2.2 Lumped-element equivalent circuit

When designing electrical circuits, it is usual to represent an inductor as an equivalent circuit of idealised lumped components; specifically, an inductance, a capacitance, and one or more resistances. To construct the model, we start by observing that, over a limited frequency range at least, the reactance presented at the terminals corresponds to that of a pure inductance in parallel with a capacitance. Then we note that there are resistive losses at all frequencies, and so we put a resistance in series with the coil. There will also be magnetic losses; e.g., core loss, and eddy-current loss in nearby conductors; but there is no need for another resistance in that case because it can be combined with the one we have already put in. Then finally, optionally, we allow that there may be dielectric losses in the wire insulation and coil former, and put a resistance in series with the capacitance.

We end up with the equivalent circuit shown on the right. This model, with suitable choice of parameters, will be found to reproduce the terminal impedance. Note however, that all of the model elements are expected to vary with frequency to some extent, and so are generally described by mathematical functions rather than simple constants.



If we measure the impedance of a coil over a range of frequencies, we will end up with a set of data that can be fitted to the model parameters. It will also be the case that R_{CL} can be set to zero for some types of coil, particularly those with air cores and some space between the turns, and three parameters will be sufficient. This does not mean however, that parameters so extracted have physical significance. Consider, for example, what might happen if we do not bother to include the frequency variation of internal inductance in the function that produces X_L. One possible justification for doing that might be that the measurements are not sufficiently accurate to determine it; but it will also be the case that there will be some frequency dependence of the other parameters, particularly X_{CL}, in which case those parameters will be adjusted in the fit to disguise the neglect of internal inductance. It follows that there is a difference between fitting the data and the extraction of physically realistic parameters, the latter being dependent on the use of realistic parameter models.

The apparent inductance, i.e., the inductance that will be found by measurement at a single frequency (after correction for lead inductance and capacitance) is given by:

$L' = X_L' / 2\pi f$

where, using the series to parallel transformation[4]:

$X_L' = [\ (R_L^2+X_L^2)/X_L\ ]\ //\ [\ (R_{CL}^2+X_{CL}^2)/X_{CL}\ ]$

R_L and R_{CL} are usually sufficiently small that the apparent inductance L' can be calculated on the basis that they are both zero. The loss resistances are however required when calculating the coil impedance (rather than just the reactance).

Recall that the hypothetical self-capacitance C_L does not predict the true SRF of the coil, but the model is reasonably accurate provided that the working frequency does not approach the SRF too closely

4 See, for example, **AC Theory**, DWK, section 19. http://g3ynh.info/zdocs/AC_theory/

2.3 Self capacitance

Formulae for the calculation of self-capacitance are discussed in another article[5].

5 **Self-resonance and self-capacitance of solenoid coils**, DWK,

2.4 AC resistance

In the design of LC matching networks and resonators, it is generally desirable to optimise the the Q of the coil for the frequency range of interest. The Q of an inductor (the component Q rather than the circuit Q) is of course, defined as the ratio of reactance to resistance, i.e.:

$Q = X_L' / R_{ac}$

where X_L' is the effective reactance (i.e., the reactance adjusted for the effect of self-capacitance and other minor dispersive effects), and R_{ac} is the frequency-dependent AC resistance of the wire. Maximising the efficiency of an inductor is therefore a matter of minimising R_{ac} ; but this requirement is complicated by the proximity effect, and by a frequency-dependent upper limit on the physical size of any coil that is to be used as a lumped inductance.

The self-resonance frequency (SRF) of a coil is dependent on the length of the wire used to wind it and the effective velocity for an electromagnetic wave travelling along the wire. The self-capacitance of the coil is our way of representing this self-resonant property using lumped components (albeit rather inaccurately). When all of the factors governing the helical propagation velocity are taken into consideration, it transpires that minimum self-capacitance is obtained when a solenoid has a length/diameter ratio of about 1, and that self-capacitance is thereafter directly proportional to the coil diameter to a very good approximation. This means that, in order to obtain a given amount of inductance for operation over a reasonably wide frequency range, it is necessary to use plenty of turns rather than a large diameter. Using turns to obtain inductance, of course, involves overlapping the conductor upon itself, and this causes the AC resistance to be greater than that dictated by the skin effect in an isolated wire. If we use thin wire to maintain some space between the turns, then we suffer from the fact that thin wires have high resistance in isolation. If we use thick wire to reduce the resistance, then we suffer from the fact that resistance is increased by the proximity effect. From all of this there arises the need to find the compromise that constitutes the optimum coil design for a particular application.

For an isolated wire at high frequencies, the AC resistance can be estimated using the following simple expressions:

$R_{\mathbf{ac}} = \rho\, \ell_w / A_{eff}$

$A_{\mathbf{eff}} = \pi\, (d\, \delta_i - \delta_i^2\,)$

$$\delta_i = \sqrt{\frac{\rho}{(\pi\, f\, \mu_{(i)})}}$$

where ρ is the resistivity, ℓ_w is the length, $A_{\mathbf{eff}}$ is the effective cross-sectional area, d is the diameter, δ_i is the skin depth, f is the frequency, and $\mu_{(i)}$ is the permeability of the wire material (assume $\mu_{(i)} = \mu_0$ for non-ferromagnetic wire). More generally however, there are accurate formulae that work at any frequency, and these are discussed in a separate article[6].

The AC resistance can also be expressed as DC resistance multiplied by a skin-effect factor Ξ (Greek upper-case "Xi"), i.e.:

6 **Practical continuous functions for the internal impedance of solid cylindrical conductors**, DWK, http://g3ynh.info/zdocs/comps/zint/ . See particularly, Section 12, the Rac-TED-ML formula, which is accurate to within 0.09%.

$R_{ac} = R_{dc}\, \Xi$

Where

$R_{dc} = \rho\, \ell_w / A_w$

$A_w = \pi\, d^2 / 4$ being the full cross-sectional area of the wire, and using the simple high-frequency formula:

$\Xi = d^2 / [4 (d\, \delta_i - \delta_i^2)]$

The DC resistance can either be measured or calculated.

In order to account for the proximity effect in coils, we can further modify the AC resistance by inclusion of a proximity factor Ψ ("Psi"), i.e., for a coil:

$R_{ac} = R_{dc}\, \Xi\, \Psi$

Ψ being defined as the ratio of the coil AC resistance to the AC resistance of the same piece of wire when not wound into a coil. This, of course, leaves us with the small problem of how to determine Ψ.

A study of solenoid coil losses was made by R G Medhurst[7]. This work has served as the basis for coil AC resistance calculations ever since, and remains useful for normal engineering purposes provided that it is applied with due regard to its limitations. Medhurst gave his results in the form of a table of Ψ values for various solenoid length / diameter and wire pitch / diameter ratios which is reproduced below. Intermediate values can be obtained by interpolation. Ψ is of course strictly frequency dependent, and the data are high-frequency limiting values, i.e., they can only be expected to give accurate results when the skin-depth is small. In practice, accuracy of better than 3% can be expected when the ratio of skin-depth to wire diameter δ_i/d is less than 1/10. For HF radio purposes, it is useful to remember that the skin-depth in copper is 50 μm at 1.75 MHz; and so, in this context, the high-frequency regime is always operative for wires of 0.5 mm diameter or greater.

Proximity factor Ψ

This gives the ratio of coil AC resistance to the AC resistance of the straightened wire, taken from Medhurst's 1947 paper (Medhurst used the symbol Φ for this factor, but since Φ is used almost universally elsewhere to represent magnetic flux, the notation has been changed here). Ψ values derived from Medhurst's empirical data are given in **bold**. Outside the top-left rectangle, Medhurst's measurements are in agreement with Butterworth's theory when the transverse magnetic-field losses are neglected, and so the Ψ values for long coils and widely spaced coils were obtained by calculation. For coils wound on formers of low-loss dielectric, the data can be expected to predict the AC resistance to better than 3% in the high-frequency regime, i.e., when $\delta_i/d < 0.1$ and the frequency is below the SRF. Strictly the values in the table are only applicable to coils having a large number of turns (i.e., $N \geq 30$). For small N, an end correction is required (see text).

p/d is the coil winding-pitch / wire-diameter ratio. ℓ/D is the solenoid length / diameter ratio.

7 **H. F. Resistance and Self-Capacitance of Single-Layer Solenoids"**, R G Medhurst (GEC Research Labs.). Wireless Engineer, Feb. 1947, p35-43; Mar. 1947, p80-92. + corresp.; June 1947, p185; Sept. 1947, p281.

p/d → ℓ/D ↓	1	1.111	1.25	1.429	1.667	2	2.5	3.333	5	10
0	**5.31**	**3.73**	**2.74**	**2.12**	**1.74**	**1.44**	**1.20**	1.16	1.07	1.02
0.2	**5.45**	**3.84**	**2.83**	**2.20**	**1.77**	**1.48**	**1.29**	1.19	1.08	1.02
0.4	**5.65**	**3.99**	**2.97**	**2.28**	**1.83**	**1.54**	**1.33**	1.21	1.08	1.03
0.6	**5.80**	**4.11**	**3.10**	**2.38**	**1.89**	**1.60**	**1.38**	1.22	1.10	1.03
0.8	**5.80**	**4.17**	**3.20**	**2.44**	**1.92**	**1.64**	**1.42**	1.23	1.10	1.03
1	**5.55**	**4.10**	**3.17**	**2.47**	**1.94**	**1.67**	**1.45**	1.24	1.10	1.03
2	**4.10**	**3.36**	**2.74**	**2.32**	**1.98**	**1.74**	**1.50**	1.28	1.13	1.04
4	**3.54**	**3.05**	**2.60**	**2.27**	**2.01**	**1.78**	**1.54**	1.32	1.15	1.04
6	**3.31**	**2.92**	**2.60**	**2.29**	**2.03**	**1.80**	**1.56**	1.34	1.16	1.04
8	3.20	2.90	2.62	2.34	2.08	1.81	1.57	1.34	1.165	1.04
10	3.23	2.93	2.65	2.27	2.10	1.83	1.58	1.35	1.17	1.04
∞	3.41	3.11	2.815	2.51	2.22	1.93	1.65	1.395	1.19	1.05

Table **4.1**

Prior to the publication of Medhurst's paper, handbook formulae for the calculation of coil resistance were usually based on the theoretical work of S. Butterworth. Medhurst showed that Butterworth's predictions are seriously inaccurate for short coils with closely-spaced turns; a problem that he attributed to faulty assumptions regarding the transverse magnetic field (i.e., the field at right-angles to the coil axis). Butterworth modelled the coil losses by assuming a uniform current through the coil and resolving the field into transverse and axial components. He then solved an infinite set of linear equations by successive approximation to determine the losses in an infinitely long solenoid. Using this as his starting point, he derived end-corrections, once again resolved into axial and transverse components, in order to modify his model to describe practical coils. The various axial and transverse field components were replaced by their RMS values before being added together to produce a table of $\Psi = R_{coil}/R_{wire}$. Medhurst's table above is a corrected version of Butterworth's table.

The problem with Butterworth's theory was that it predicted infinite losses for coils with closely spaced turns, and unrealistically high losses otherwise, *except* for the case of a very long coil. Medhurst observed that the transverse field disappears when the coil is infinitely long (think of the field-lines around a very long bar-magnet - all are parallel to the axis), and so deduced that the errors were due to an excessive contribution from the transverse field. Nowadays, we might also observe that the high-frequency properties of an inductor are best deduced by consideration of electromagnetic waves travelling along the wire, and using the notion of energy in transit, it is obvious that an accurate description of the coil behaviour requires Maxwell's equations, not magnetostatics. Qualitatively this implies that, for a given propagation mode, the magnetic field at any point is locked at right angles to the electric field; and so, notwithstanding the numerous approximations used, there are constraints on the field pattern beyond those envisaged in Butterworth's theory. Medhurst sidestepped this problem by dropping the transverse magnetic component completely, greatly increasing the range over which the calculated losses were in agreement with experiment. For the area in which disagreement remained, i.e., the top left rectangle in table **4.1** above, he filled in the table with experimentally obtained values. Interestingly, from all of this, we can deduce that the principal electromagnetic propagation mode in a long solenoid coil has its *electric* vector very-nearly perpendicular to the coil axis.

The Medhurst-Butterworth proximity factor provides a quick and easy method for estimating the losses in coils operating in the high-frequency regime, but it is by no means the whole story. There is, in particular, an unquantified frequency-dependence as a coil passes from the low-frequency to the high-frequency regime. On this matter, it is interesting to note that the rearrangement of field patterns that occurs in this interval is related to the change in effective current-sheet diameter from D_0 to D_∞. In the author's article on internal impedance, it was observed that the skin-effect and the internal inductance of a wire are derived from the real and imaginary parts of the internal impedance. Here we should note that the proximity effect also has real and imaginary parts, and the proximity factor and the frequency dependence of the effective current-sheet diameter are thus related. It should therefore be possible to predict the high-frequency inductance from the proximity factor (or vice versa). The author is unaware of any rigorous formula enabling this to be done (i.e., a theoretically justified expression for the elusive D_∞ in terms of Ψ), but it is possible that Butterworth's approach of resolving the problem into a long-coil formula with end corrections will work. We might also make the pragmatic observation that just about any function that moves the effective diameter away from D_0 and towards D_0-d (i.e., the inside diameter) as Ψ increases will improve the accuracy of an inductance calculation.

A more recent study of solenoid AC resistance was given by Fraga et al[8]. This was a theoretical investigation applicable to the long solenoid case, presenting some difficulty in its integration with the techniques discussed here. An interesting outcome however, is that the skin effect and the proximity effect are not theoretically separable. To treat the subject correctly, the skin effect must be replaced by a modified skin effect which includes the proximity effect. The implication is that the two factors which modify the DC resistance act in concert and give rise to a single dispersion region; i.e., the onset of the high-frequency resistance regime, and the associated drop in inductance, occur in the same part of the spectrum for both skin and proximity effects.

End correction:

The Ψ values given in table **4.1** are strictly applicable only in the case where the number of turns in the coil, N, is large. Medhurst offered a tentative but plausible end-correction for coils with less than 30 turns, which can be understood as follows.

For a coil with a large number of turns we can write a formula for the AC resistance of a single turns as:

$$R_{ac} / N = R_{dc}\ \Xi\ \Psi / N$$

This expression can be taken to be true for turns in the middle region of any coil, but for the two turns at the ends of the coil, which lack an adjacent turn on one side, the proximity effect will be reduced by about a factor of 2. Hence, we should think of a coil as being made up of N-2 turns subject to the full proximity effect, and 2 turns subject to half the proximity effect of the others. Thus the expression for AC resistance becomes:

$$R_{ac} = R_{dc}\ \Xi\ [\ (N-2)\Psi + 2\Psi/2\ ]/N$$

i.e.,

$$R_{ac} = R_{dc}\ \Xi\ \Psi\ (N-1)/N \qquad (\mathbf{4.1})$$

8 **Practical Model and Calculation of AC resistance of Long Solenoids**. E. Fraga, C Prados, and D.-X Chen. IEEE Transactions on Magnetics, Vol 34, No. 1. Jan 1998.

The end correction (N-1)/N will make practically no difference if N is large, and so (**4.1**) becomes the preferred expression for the AC resistance of a coil.

Asymptotically correct formula for frequency dependence of proximity factor:
Modelling R_{ac} across the dispersion region.

As is also the case for internal inductance, the transition from low to high frequency behaviour occurs when $d/2 = r_w = \delta_i$.

It is reasonable to assume that when $\Xi = 1$, $\Psi_{\mathbf{eff}} = 1$.

So, we need to weight the influence of the Ψ (N-1)/N factor according to Ξ.

As a first attempt at this we can write:

$R_{\mathbf{ac}} = R_{\mathbf{dc}} + R_{\mathbf{dc}} (\Xi -1) \Psi (N-1)/N$

This is is an improvement over simple multiplication, but the (N-1)/N factor causes R_{ac} to be underestimated when Ψ=1.

To correct that, we can replace N-1 with N-(Ψ-1)/Ψ.

Hence:

$R_{ac} = R_{dc} + R_{dc} (\Xi -1) \Psi (N-1+1/\Psi) / N$

i.e.:

$R_{ac} = R_{dc} [1 + (\Xi -1) \Psi (N-1+1/\Psi) / N]$

This is doubly asymptotic (i.e., correct at both low and high frequencies), and so is the best we can do with Medhurst's data.

Paul Zwicky has used this calculation method and found good agreement with experiment[9].

9 **Optimierung der Güte einlagiger zylindrischer Luftspulen**, Paul Zwicky HB9DFZ, Funkamateur, Okt. 2013, p1080-1084, + picture on P1032. English translated work files relating to that article are given at: http://g3ynh.info/zdocs/magnetics/appendix/optQ_hb9dfz.html