On a modified form of Wheatstone's bridge and methods of measuring small resistances

by George Carey-Foster (1835 - 1919)

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On a Modified form of "Wheatstone's Bridge," and Methods of Measuring Small Resistances, by G. C. Foster, F.R.S., Professor of Physics in University College, London.

The object of this paper is to describe simple methods for the following purposes connected with the measurement of electrical resistances, namely :—

I. The measurement of small resistances, by ascertaining what length of a graduated standard wire has an equal resistance;

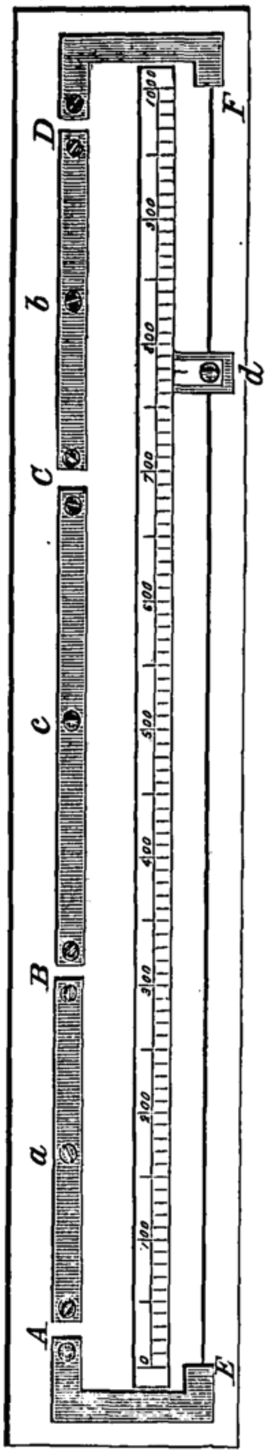
II. The electrical 'calibration' of wires, or the division of them into lengths of equal resistance;

III. The elimination of errors due to the resistance of connections from determinations of specific resistance.

It will be seen from what follows, that the arrangement of apparatus employed for these purposes might be varied in many ways, but it will probably conduce to clearness if I confine my description to the arrangement that I have actually used. In all the experiments, the principal piece of apparatus, besides a battery and a galvanometer, was a Differential Resistance Measurer,—or, as it is commonly called, a "Wheatstone's Bridge,"—of the form usually made by Messrs. Elliott Bros., for purposes of demonstration and research; and, as I shall have to make frequent reference to this instrument, it may be well to give a brief description of it before speaking of the special purposes to which it has been applied.

Figure 1 shews the arrangement of its essential parts on a scale of about $\frac{1}{9}$. A piece of german-silver wire, E F, from 1·5 mm. to 2 mm. in diameter and 1 metre long, is stretched upon an oblong board forming the base of the instrument, parallel to a metre-scale divided throughout its whole length into millimetres, and is so placed that its two ends are approximately opposite to the divisions 0 and 1,000 respectively of the scale. The ends of the wire are soldered to a broad copper band, which passes round each end of the graduated scale and runs parallel to it on the opposite side from the wire. This band is interrupted by four gaps, A, B, C and D, at each side of each of which is a binding screw for connecting the conductors whose resistance is to be measured. In the ordinary use of the apparatus, the wires from the battery are attached to the screws *a* and *b* fixed upon the copper strips A B and C D respectively, and the galvanometer wires are connected, one to the screw *c* on the strip B C, and the other to the screw *d* upon a moveable block, which slides along the graduated scale and allows contact to be made at any part of the german-silver wire, while an index-mark shows the distance of the point of contact from each end of the scale. The conductors to be compared are inserted at the two middle gaps B and C, and if they are of small resistance the end gaps A and D are usually closed by thick copper latches, or, if the resistances at B and C are considerable, conductors, whose resistances are known in terms of that of the wire E F, are inserted at A and D; it is easy to see that the effect of these, which may be regarded as ungraduated prolongations of the german-silver wire, is to increase the delicacy but to limit the range of the instrument. When the moveable contact *d* has been so placed that, on completing the battery-circuit, the galvanometer shows no deflection, it follows from the general principle on which the measurement is founded that—

FIG. 1.



resistance a to c : resistance c to b = resistance a to d : resistance d to b.

In most cases we may substitute for this, as a fair approximation,—
resistance at B : resistance at C = resistance E to d + resistance at A : resistance d to F + resistance at D, or

$$= \text{scale reading} + \left\{\begin{matrix}\text{length of wire equal}\\ \text{in resistance to A}\end{matrix}\right\} : 1{,}000 - \text{scale reading} + \left\{\begin{matrix}\text{length of wire}\\ \text{equal in}\\ \text{resistance to D}\end{matrix}\right\};$$

but in adopting this last value as representing the ratio of the resistances at B and C, we are exposed to sources of error which need not be pointed out to any one accustomed to electrical measurements, and which become of greater importance in proportion as the resistances to be compared are smaller. These errors are almost entirely avoided in the process now to be described.

I.—*Method of measuring small resistances by comparison with a graduated standard wire.*

The resistance of any wire, whose resistance is less than that of the whole length of graduated german-silver wire, E F, may be found in terms of the latter as follows:—Insert the wire to be measured at the gap A, close the gap D by a conductor of insensible resistance, and insert at B and C any two convenient conductors the ratio of whose resistances (which it is not necessary to know) does not differ from unity more than does that of the resistance to be measured and the resistance of the whole wire E F; then shift the moveable contact d until the galvanometer ceases to be deflected, and take the reading of the scale. Next put the wire to be measured at D and close the gap A by a conductor without sensible resistance; shift d until the galvanometer is again balanced, and take a second reading of the scale: *the difference of the two scale-readings gives the length of the wire* E F *whose resistance is equal to that of the wire to be measured.*

This result is almost self-evident, but it may be proved, if needful, thus:—Let L denote the total resistance of the wire E F, A and D, the resistances inserted at the two end gaps, e and f the resistances of the copper bands between a and E and between b and F respectively, and let B and C stand for the total resistances between a and c, and between c and b respectively; also let m be the scale-reading when the resistances A and D are in their first positions, and m' the reading when they have been interchanged; lastly, let k be the resistance of unit length (1 millim.) of the wire E F. Then the two experiments give—

$$\frac{B}{C} = \frac{A + e + mk}{L - mk + D + f} \text{ and } \frac{B}{C} = \frac{D + e + m'k}{L - m'k + A + f},$$

whence, by equating the two values of $\frac{B}{C}$ and adding unity to each side of the equation, we get:—

$$\frac{A + D + L + e + f}{L - mk + D + f} = \frac{A + D + L + e + f}{L - m'k + A + f},$$

or

$$A - D = (m' - m)k.$$

In the measurement described above, we assumed the condition $D = 0$; if m_0 and m'_0 be the scale-readings corresponding to this condition, we have—

$$A = (m'_0 - m_0)\,k.$$

But, since in practice the resistance D can never be reduced absolutely to nothing, it is important, when great accuracy is required, that its value should be known: a method is given further on (p. 203, 204) by which any resistance great enough to affect sensibly the reading of the instrument can be measured.

In order that measurements made in the manner that has been described may have a definite meaning, it is of course necessary that the value of the coefficient k, or the resistance of unit length of the wire E F should be known with reference to some recognised standard; and when, as in the instrument I have chiefly used, the resistance of the whole wire E F is a comparatively small fraction (about $\frac{1}{7}$) of a unit, some special method is needed for measuring k.

I have employed two methods for this purpose: the one which is, on the whole, the most convenient being the following. A wire whose resistance, R, is only a little less than that of the whole wire, E F, is measured in terms of the latter by the process given above: let p be the number of millimetres between the two readings of the positions of the sliding contact, then

$$k\,p = R \ldots\ldots (1)$$

Next a standard (unit) coil of resistance S is combined in multiple arc with the wire already measured, and the measurement is repeated: this gives, if q be the number of millimetres between the two readings in this case,—

$$k\,q = \frac{R\,S,}{R + S} \ldots\ldots (2)$$

Whence we get, as the value of the coefficient required,

$$k = S\frac{p - q}{p\,q}.$$

Another method is to insert a standard coil at one of the gap ends (say at A), and at the other (D) a wire whose resistance R_1 falls short of that of the standard by not quite the whole resistance of E F, and to shift the moveable contact until the galvanometer is balanced; then to interchange the standard and the resistance R_1 and shift the contact again till the balance is restored: let d be the difference (in millimetres) between the two readings of the scale in this experiment. Next take a second wire whose resistance, R_2, falls short of that of R_1 by nearly the whole resistance of E F, and proceed with this and the first wire in the same way as with the first wire and the standard, and let d_1 be the difference between the two readings of the scale in this case. Proceed in this manner with wires of smaller and smaller resistance until one is arrived at of smaller resistance than the wire E F. Let the resistance of this wire be R_n: by inserting it at one of the end gaps and an insensible resistance at the other, and afterwards interchanging, we find by taking the difference, d_n, of the scale-readings in the two positions required to balance the galvanometer, the number of millimetres of the graduated wire whose resistance is equal to R_n. By a set of such experiments we obtain—

$$\begin{aligned} d\,k &= S - R_1, \\ d_1 k &= R_1 - R_2, \\ &\vdots \\ d_{n-1} k &= R_{n-1} - R_n, \\ d_n k &= R_n; \end{aligned}$$

or

$$k = S \; \frac{1}{d + d_1 + \ldots + d_{n-1} + d_n}.$$

II.—*Method of testing the resistance of the various portions of a conducting wire, and dividing it into parts of equal resistance.*

It has been tacitly assumed in the foregoing that the coefficient k, or the resistance of 1mm. of the graduated wire E F, is constant from end to end. This, of course, is never strictly the case, and hence it is desirable to measure not only the mean value of this coefficient for the whole wire, but also its value for the different parts, or else to assure ourselves that the variations of value are too small to be of importance. This examination of the wire is very readily made by the arrangement represented in Fig. 2, which shews a second graduated wire E′ F′ connected with a "Bridge" of the form already described. So far as the same reference-

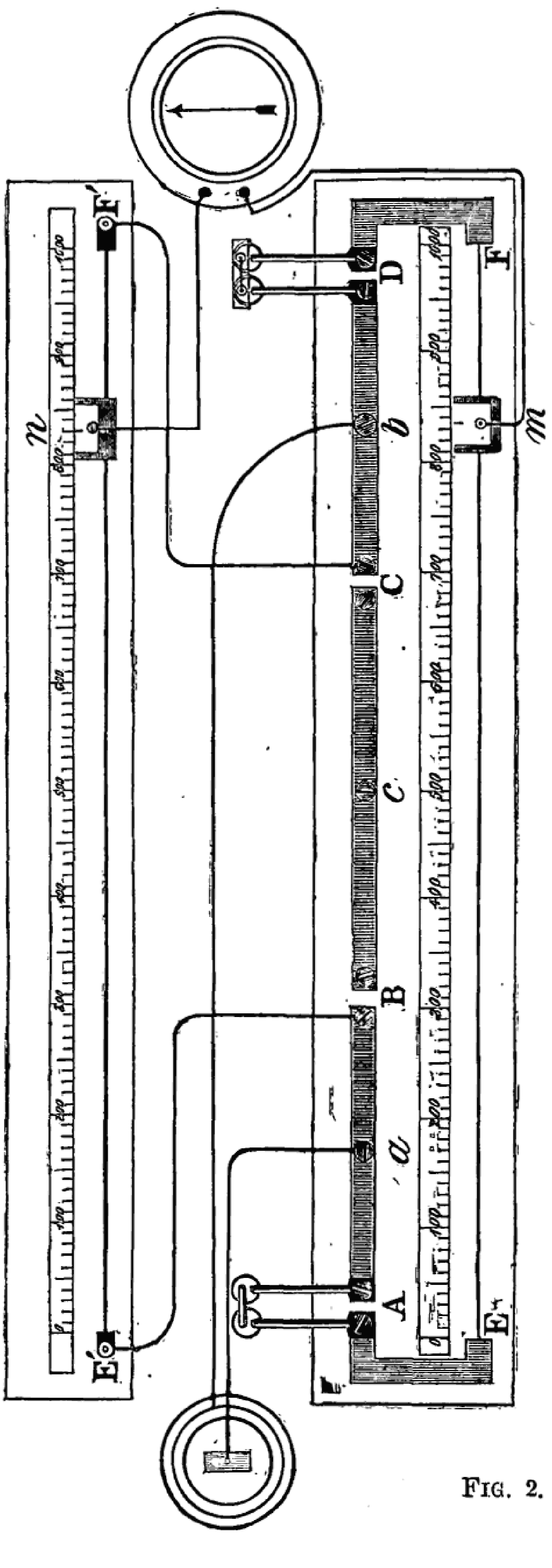


Fig. 2.

letters occur in this figure and in Fig. 1, they mark identical parts: the battery is connected (through a make-and-break key not shewn in the diagram) with the binding-screws *a* and *b*, and the terminals of the galvanometer are connected with *two* moveable contact-makers, one of them *m* on the principal wire E F, and the other *n* on the wire E′ F′—for convenience of reference this will be afterwards called the "compensating wire" or "compensator"—the ends of which are connected with the outer binding-screws of the gaps B and C. The end gaps A and B are closed, the one by a short and very thick piece of copper wire of almost insensible resistance, and the other by a short piece of german-silver wire of resistance equal to that of whatever length of the graduated wire it is desired to test at once. This piece of wire serves the purpose of a gauge with which each part of the wire E F is compared in succession: in most of my experiments its resistance was equal to that of about 70 mm. of the wire to be tested. The gauge being (say) at the right-hand gap D (as in the diagram) the sliding contact *m* is placed at or very near to the right-hand end of E F, and the slider *n* is moved on the compensating wire so as to destroy the deflection of the galvanometer. The gauge and the thick copper connector are next interchanged, and the balance is restored by moving the slider *m*: as already pointed out (p. 198) the length of E F over which the slider is moved, is that of which the resistance is equal to the difference between the resistances of the gauge and the thick connector. The gauge and connector are now put back into their first positions, and the slider *n* is shifted until the galvanometer is again balanced; then they are interchanged once more and the balance obtained by shifting the slider upon E F. This process of successively interchanging the gauge and connector, and adjusting the galvanometer to zero, by moving alternately the sliders *m* and *n*, is continued until the former has been brought step by step to the further extremity of the wire E F. It will be seen that each shift of the sliding contact *m* is over a portion of the wire E F whose resistance is equal to the constant difference between the resistances of the guage and connector, and therefore that, by the process that has been described, the graduated wire is divided into parts of equal resistance, in a manner exactly comparable to that in which a thermometer-tube is "calibrated," or divided into portions of equal capacity, by marking on it the lengths successively occupied by a small quantity of mercury which is pushed along it.

The shift of the slider *n* also takes place over portions of the wire E′ F′ which have all an equal resistance, but this bears the same ratio to the resistance represented by the shifts of the slider m that the resistance of the branch a B E′ F′ C b bears to the resistance of the branch a A E F D b. To prove this, let the sliders be so placed that the galvanometer is in equilibrium when the gauge is at the gap D, and the gap A is closed by the thick connector: let the resistance of the gauge be denoted by G, that of the connector by C, the resistances of the wires E F and E′ F′ by L and L′ respectively, those of the permanent connections between a and E and between a and E′ by e and e', the permanent resistances between b and F and between b and F′ by f and f', and lastly the resistances of E m and E′ n by r and r', We then have the equation—

$$\frac{C + e + r}{G + f + L - r} = \frac{e' + r'}{f' + L' - r'} \quad \ldots\ldots \quad (1)$$

Then, by interchanging the gauge and connector, and balancing the galvanometer again by moving the slider n nearer to F′, while the slider m remains where it was, we get the equation—

$$\frac{G + e + r}{C + f + L - r} = \frac{e' + r'_1}{f' + L' - r'_1} \quad \ldots\ldots \quad (2)$$

where r'_1 is the resistance between E′ and the new position of n, and therefore $r'_1 - r'$ is the resistance of that portion of the compensating wire over which the slider has been moved. By adding unity to each side of each of the equations (1) and (2), inverting the two new expressions thus obtained, and subtracting one from the other, we get—

$$\frac{r' - r'}{e' + f' + L'} = \frac{G - C}{G + C + e + f + L}$$

or

$$r_1' - r' = (G - C)\,\frac{e' + f' + L'}{G + C + e + f + L}; \quad \ldots\ldots \quad (3)$$

which shews that, with the same apparatus, at each shift of the moveable contact along the compensating wire, it passes over a constant resistance of the magnitude stated above.

The value of the multiplier of G—C in equation (3) is easily found experimentally; for, since the resistance passed over at each shift of the slider on the principal wire, E F, is equal to the difference G—C, denoting this resistance by $r_1 - r$, we obtain—

$$\frac{e' + f' + L'}{G + C + e + f + L} = \frac{r'_1 - r'}{r_1 - r} = M.$$

By help of this value, which for shortness may be denoted by M, exceedingly small resistances, such as that of the short copper connector mentioned above, are readily measured when they are connected so as to form part of the principal circuit. For this purpose, it is sometimes needful (in order that every measurement may come within the range of the instrument), to transfer the battery-wires from the binding-screws a and b to E′ and F′; when that is done, the factor M must be determined in this state of the apparatus, since the resistances e' and f' now belong to the principal branch of the circuit, instead of to the compensator branch. The galvanometer wire is now disconnected from the slider m and applied to one end of the connector (or other conductor whose resistance is to be measured), and the balance is got by means of the slider n; the galvanometer wire is next applied to the other end of the connector, and the balance is again got by the slider. Then if Q be the resistance over which the slider is moved, the resistance of the connector is given by—

$$C = \frac{Q}{M}.$$

As an example of the resistances which are easily measurable in this manner, it may be mentioned that in two experiments the resistance of a thick and short piece of copper wire was found to be equal to that of 0·6 millimetre of the german-silver wire E F, or to be about 0·00008 of a B. A.-unit.

When the graduated wire of the bridge has been divided as above described into sections of equal resistance, it is of course easy to make a table showing what fraction the resistance of any given part of it is of the total resistance. The manner of doing this is too obvious to need further description For the sake of comparison, a table was made by the above process for a bridge-wire which had become very irregular through rough usage, and a similar table was made for the same wire from the results obtained in comparing together by means of it, in the usual way, a set of coils of known resistances.* The numbers obtained

* The following was the method of calculating the resistance of each part of the wire from the experimental numbers :—

Let L denote the total resistance between the battery-connections through the bridge-wire, that is to say, the resistance of the branch a A E F D b (Fig. 1).
,, R the resistance inserted at the left-hand centre gap B.
,, r ,, ,, right-hand ,, C.
,, x the (unknown) resistance between the zero-end (left-hand end) of the bridge wire, and the nearest battery contact, that is, the resistance from B through A to a.
,, a the scale-reading for given values of R and r; and
,, k the mean resistance of 1 mm. of the bridge-wire between zero and the point a.

Further let the values of the above symbols corresponding to the smallest value given to the ratio R : r be distinguished as R_0, r_0, a_0, k_0, and those corresponding to the greatest value of this ratio be distinguished as R_n, r_n, r_n, k_n.

Then any two experiments in which intermediate values are given to the ratio R : r give the two equations—

$$\frac{R_p}{r_p} = \frac{a_p k_p + x}{L - a_p k_p - x} \quad \text{and} \quad \frac{R_q}{r_q} = \frac{a_q k_q + x}{L - a_q k_q - x};$$

by the two processes were almost identical; but I consider, nevertheless, that the method of the compensating wire and gauge possesses some advantages, since it does not depend upon the accuracy with which a set of resistance-coils have been previously adjusted, and the numerical calculations required are so simple that there is little danger of arithmetical error.

III.—*Method of avoiding errors due to the resistance of connections in determinations of specific resistance.*

A method for this purpose has been described by Sir William Thomson (*Proc. Roc. Soc.*, [1861] vol. xi., p. 313), but probably a simpler and equally accurate process is to connect the wire to be used for the determination, in the place of the german-silver wire E F of a bridge of the form shewn in figure 1, and to proceed precisely in the manner already indicated at page 199, for the measurement of the resistance of the unit length of the bridge-wire. The result is in this case wholly independent of any resistance outside the extreme positions of the sliding contact-maker, and since this is only used for finding a point which gives *no current* through the galvanometer, the result is also unaffected by any moderate variation of resistance at the moveable contact itself. I do not think there could be any great difficulty in arranging the apparatus so as to admit of the wire between E and F being immersed in a bath of liquid, so that its temperature might be known with greater accuracy than is possible when it is exposed to the air, but I have made no attempts in this direction.

It will be seen that the accuracy of the methods of measurement

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whence we get, for the resistance of the wire between the points a_p and a_q,—

$$a_q\,k_q - a_p\,k_p = L\frac{R_q\,r_p - R_p\,r_q}{(R_p + r_p)\,(R_q + r_q)} \quad \ldots\ldots \quad (a)$$

Similarly, for the resistance of the whole wire, or more accurately for the portion of it between the extreme readings a_o and a_n, we have—

$$a_n\,k_n - a_o\,k_o = L\frac{R_n\,r_o - R_o\,r_n}{(R_o + r_o)\,(R_n + r_n)} \quad \ldots\ldots \quad (b)$$

The left hand side of this equation may also be written $(a_n - a_o)\,k'$, where k' stands for the average resistance of 1 mm. for the whole length of the wire, a quantity which can be measured in the manner already mentioned (p. 199—200); hence the elimination of L from equation (*a*) by means of equation (*b*) gives—

$$a_q\,k_q - a_p\,k_p = (a_n - a_o)\,k'\,\frac{(R_o + r_o)\,(R_n + r_n)}{R_n\,r_o - R_o\,r_n}\cdot\frac{R_q\,r_p - R_p\,r_q}{(R_p + r_p)\,(R_q + r_q)}$$

It will be noted, that only the last factor of this expression varies from one experiment to another of any one set, and hence the remainder can be calculated once for all.

described in this paper, depends essentially on the possibility of connecting the conductors inserted at the two gaps A and D with the rest of the apparatus in such a way, that they can be taken out and put back again time after time, without causing any perceptible change in the resistance of the connections. In this respect, I have found the use of well-amalgamated copper rods, resting by their flat ends on an amalgamated plate of copper, forming the bottom of a mercury-cup of the form devised (I believe) by the late Dr. Matthiessen, and described by him and Mr. Hockin (*Rep. Brit. Assoc.*, 1864, p. 354), quite satisfactory.

DISCUSSION ON THE PAPER.

THE PRESIDENT having called upon members to ask any questions, or offer observations on the paper,

Mr. LATIMER CLARK said he was sure they had all listened with pleasure and advantage to the paper just read. To his own mind it seemed to add one more proof of the great value of the system of measuring by the Wheatstone Bridge, which was so indispensable to electricians. It might surprise some to hear that the instrument Prof. Foster had spoken of was not the invention of Wheatstone; yet it was in fact, invented by Mr. S. Hunter Christie, who read a paper upon it before the Royal Society in 1833, ten years before Prof. Wheatstone brought it a second time before the notice of the public. Mr. Christie was attached to the Royal Military Academy at Woolwich, and was the author of many papers on electricity in the *Philosophical Transactions*, and, among others, of an article on this subject which was dated 28th February, 1833, and in which he described what he termed his "differential arrangement," and the manner in which the instrument was applied to the measuring of the relative conduction of different metals. Among other things he compared the resistance of a length of 350 inches of No. 16 gauge wire, with a length of No. 22 gauge, which required 90 inches to balance it. Mr. Christie also employed it as a means of discovering the conducting powers of metals at different temperatures, and his arrangements were so sensitive that it was observed that the warmth of the hand causes the needle to vary sensibly. It was deduced, from a number of experiments, that the conducting power of metals varied as the square of the diameter, a fact which was not so well known then as now. Notwithstanding all the endeavours that were made at the time to make the invention known, it remained in oblivion till Sir Charles Wheatstone brought it forward again in the year 1843, when he wrote his remarkable paper on electrical measurement, in which he stated the source from which this invention was derived, and gave the dates of the papers which Christie had written on the subject. It was, therefore, not from any fault of Sir Charles Wheatstone that his name became attached to the instrument, although it was only fair to say that Christie did all in his power to make it known, and to point out its advantages. It was often the case that a man might be too soon as well as too late: in those days telegraphy was almost unknown: but in 1843, when Wheatstone called attention to the instrument, telegraphy had commanded great attention. Moreover, he had not long before carried out his brilliant experiments on the measurement of the velocity of light, which were so well known, and from that cause, combined with the admirable character of the paper in which the instrument was re-described, it had been since known as the Wheatstone Bridge.

Mr. S. Phillips, Junr., asked whether, in measuring small resistances, the functions which the resistance of the galvanometer might have upon them, had been investigated? He would like to be informed what was the resistance of the galvanometer, and what would be the most suitable resistance in order to get the greatest sensibility.

Professor Foster replied that he had not specially investigated that point. What he had actually used was a resistance of about 1 unit in the galvanometer. This matter was fully gone into, but he did not remember the results, some years ago by Mr. Schwendler in the *Philosophical Magazine*, which he thought was the best authority to refer to on the subject.

The President remarked that the resistance of the galvanometer should be nearly equal to the resistance to be measured, and the maximum of magnetic momentum was got, with arms of the balance of equal resistance.

Mr. Latimer Clark described by means of a diagram on the board, the form in which the instrument was used, as given in Mr. Christie's paper. He added, he should have reminded the meeting how much they were indebted to Dr. Werner Siemens for the great improvements he had effected in the instrument with respect to the unequal proportions of measurement, it had added tenfold to the value of the bridge.

The President said, having been connected more or less with the Wheatstone Bridge in its early application to telegraphic measurement, he would say that the instrument was first attempted to be used in testing coils of insulated line-wire by his brother in 1847-48. It was soon found that the range of the instrument was insufficient, and it occurred to his brother to construct these resistance-boxes so as to make the two arms of the balance variable, which gave a much larger range of reading. Instead of simply adjusting one weight with another on equal arms of the balance he made so to speak, the length of the arms variable, and got thereby much wider limits within which the instrument could be applied. Moreover, by the adoption of resistance-boxes a great deal of the difficulty which Professor Foster had met with was avoided; because the stopper made a very safe contact. Moreover the wire with sliding contact was apt to wear if much used, and the resistances of comparison consisted of fractions of units only, whereas the resistance to be measured amounted usually to hundreds, thousands, hundreds of thousands, and millions; it also followed from this condition that a galvanometer of exceedingly small resistance must be used, in order to get a proper proportion between those resistances, to give a maximum effect; whereas with box-resistances they could use a galvanometer of many thousand units in its point of maximum sensitiveness. He would ask Professor Foster to be good enough to state whether he had compared his improved instrument—which appeared to be very ingenious, as a method of avoiding errors in dealing with small resistances—whether he had compared that with the instrument with the resistance boxes?

Professor Foster replied that his mode of comparing resistances was one which would usually be adopted only where the resistances to be compared were very small, and for that purpose he suggested it. If they had to measure the resistance of two or three feet of moderately thick copper wire, then the resistance which was introduced even at the best made stoppers was a thing which could not be neglected. In measuring high resistances it was easy to make this resistance unimportant, but in dealing with small amounts it was not so easy; and for that reason he had not compared measurements made in the way that he had described with those obtained by means of resistance coils. One method was applicable to low resistances, and the other to cases of high resistance. Another difficulty with the resistance coils was they could only go by one unit at a time; though by using an unequal arm balance they could subdivide by hundreds and thousands, but then the measurement became less accurate. They introduced great resistance on one side and small on the other.